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Symplectic Geometries, Transvection Groups, and Modules

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We show that any connected partial linear space in which there is a line with at least four points and that has the property that any pair of intersecting lines is contained in a subspace isomorphic to a symplectic plane is isomorphic to the geometry of hyperbolic lines in some symplectic geometry. As a corollary to this result we obtain a characterization of the subgroups of the symplectic groups that are generated by transvection subgroups. Also a characterization of the natural modules for these groups is obtained.

1. INTRODUCTION

A *symplectic plane* (also called a *dual affine plane*) is a projective plane from which a point and all the lines through that point are removed. In this paper we study those partial linear spaces in which any pair of intersecting lines is contained in a subspace isomorphic to a symplectic plane. Such spaces have been studied by several people, see for example [4–12, 14], but it was Hall who obtained the most complete results on such geometries. In particular, Hall shows in [7, 11] that if such a space is finite, connected, and contains a line with at least four points, it is isomorphic to the geometry of hyperbolic lines in a symplectic geometry. Hall also expresses the belief that this is true without the finiteness assumption. It is the purpose of this paper to show that this is indeed the case. We prove:

THEOREM 1.1. *Let (P, L) be a connected partial linear space in which any pair of intersecting lines is contained in a subspace isomorphic to a symplectic plane. Assume that (P, L) contains at least two such planes and a line with more than three points. Then (P, L) is isomorphic to the geometry of hyperbolic lines of a symplectic geometry embedded in some Desarguesian projective space of dimension at least* 3.

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If one allows lines to have three points then the above result is not true anymore. In [9, 10] Hall obtains a complete classification in this particular case. Although not all examples are symplectic they all can be embedded as a subspace in a geometry of hyperbolic lines of a symplectic space over $GF(2)$.

Our proof of the above result is completely geometric and elementary. This is in contrast to Hall's approach in the finite case, where the proof is divided into a geometric and a group theoretical part. We obtain some group theoretical results as corollaries to Theorem 1.1.

Let G be a group and k a division ring. Suppose Σ is a conjugacy class of subgroups of G such that the following hold:

1. $G = \langle \Sigma \rangle$;
2. for all $A, B \in \Sigma$ we either have that $[A, B] = 1$ or $\langle A, B \rangle \simeq (P)\, SL(2, k)$ and A and B are full unipotent subgroups of $\langle A, B \rangle$.

Then Σ is called a *class of k-transvection subgroups of G*. See [15]. If we also have that

3. for all A, B and $C \in \Sigma$ with $[A, B] \neq 1 \neq [A, C]$ the group $\langle A, B, C \rangle$ is either contained in $\langle A, B \rangle$ or is modulo its center isomorphic to a split extension of the dual of the natural two-dimensional $SL(2, k)$-module by $SL(2, k)$,

then Σ is called a *class of k-transvection subgroups of symplectic type* in G.

THEOREM 1.2. *Suppose k is a division ring with at least three elements. Let G be a centerfree group generated by a conjugacy class Σ of k-transvection subgroups of symplectic type. Then there is an isomorphism from G into $TSp(V, f)$ for some k-vector space V and nontrivial symplectic form f (or $PSp(V, f)$ when V has finite dimension and f is nondegenerate) or into $T(R, V' \backslash U)$ for some k-vector space V' with subspace U of codimension 2 and $R = Ann_{V'^*}(U)$. Moreover, this isomorphism maps the elements of Σ onto the transvection subgroups of $TSp(V, f)$ (or $PSp(V, f)$) or $T(R, V' \backslash U)$.*

For the notation the reader is referred to Section 6. The main difference between the known characterizations of the symplectic groups as groups generated by their transvection subgroups, see for example [1, 11, 15], is that we impose no finiteness conditions and allow nontrivial solvable normal subgroups.

Suppose the symplectic space (V, f) in the above theorem is degenerate and W is a complement in V to the radical of f. Then the group $TSp(V, f)$ is a split extension of a direct sum of copies of W by the group $FSp(W, f|_W)$. This observation leads to the following characterization of the natural modules for the symplectic groups (or their subgroups generated by transvection subgroups).

THEOREM 1.3. *Let* (V, f) *be a nondegenerate symplectic space over the field* k. *Suppose* k *contains at least three elements and* V *has dimension at least* 4. *Let* G *be the group* $FSp(V, f)$ *and* Σ *the class of symplectic transvection subgroups of* G.

If M *is a nontrivial* $\mathbb{Z}G$*-module such that*

1. $C_M(G) = 0$,
2. $[M, G] = M$,
3. $[M, A, B] = 0$ *for all* $A, B \in \Sigma$ *with* $[A, B] = 1$,

then M *is a direct sum of natural* kG*-modules isomorphic to* V.

In Hall [11] and Timmesfeld [16] one can find similar results under some finiteness conditions. In the same vein Meierfrankenfeld [13] has given module characterizations for the subgroups of all the classical groups generated by their long root subgroups. The methods used in these module characterizations are of an algebraic nature. Our proof of Theorem 1.3 depends on Theorem 1.1 and hence is mainly geometric.

In the remainder of this Introduction we give a short outline of this paper. In Section 2 we give some definitions and discuss the symplectic geometries appearing in the conclusion of Theorem 1.1. Sections 3 to 5 are devoted to the proof of our main result, Theorem 1.1.

Let ∞ be a fixed point of a partial linear space (P, L) as in the hypothesis of 1.1. Then consider the subset of P consisting of all points in P different from but collinear with ∞. We shall define a structure on this subset of P that makes it into an affine space. The natural embedding of this affine space into a projective space supplies us with an embedding of (P, L) into a projective space equipped with a nontrivial symplectic polarity, such that L is the set of hyperbolic lines of that projective space with respect to the polarity. This finishes the proof of 1.1.

In Section 6 we discuss some properties of symplectic transvection groups needed for the proofs of the Theorems 1.2 and 1.3. These are given in Sections 7 and 8, respectively.

2. SYMPLECTIC GEOMETRY

A *partial linear space* (P, L) is a set P whose elements are called *points* and a set L of *lines* which are subsets of P such that every line contains at least two points and every pair of points is in at most one line.

Let (P, L) be a partial linear space. If p and q are two distinct points of P then we say that they are *collinear*, notation $p \sim q$, whenever there is a line in L containing both p and q. (Note that a point is not collinear with itself.) If p and q are collinear points then pq denotes the unique line containing both p and q. If p is a point then by $p^{\sim}$ we denote the set of points collinear with p. The *collinearity graph* of (P, L) is the graph with vertex set P and two vertices adjacent if and only if they are collinear. The space (P, L) is called *connected* if and only if its collinearity graph is connected.

A *subspace* of (P, L) is a subset P' of P with the property that whenever p and q are two collinear points of P' that are on the line $l \in L$, then $l \subseteq P'$. If P' is a subspace of (P, L) then P' together with the set of lines in L that meet P' in at least two points forms a partial linear space. We often identify P' with this partial linear space. It is clear that the intersection of any collection of subspaces is again a subspace. Therefore we can define for any subset X of P the subspace *generated by* X and denoted by $\langle X \rangle$ to be the intersection of all subspaces containing X. A *plane* of (P, L) is a subspace generated by two lines that intersect nontrivially. A *hyperplane* of (P, L) is a subspace of (P, L) that meets every line nontrivially. A hyperplane is called a *proper hyperplane* if it is different from P.

Let $\mathbf{P}$ be a projective space. A polarity $\perp$ of $\mathbf{P}$ is a map from the set of points of $\mathbf{P}$ into the set of hyperplanes of $\mathbf{P}$ mapping a point p to $p^{\perp}$ such that:

1. $p \in q^{\perp}$ if and only if $q \in p^{\perp}$.

The polarity is called *symplectic* if we have furthermore

2. $p \in p^{\perp}$ for all points p of $\mathbf{P}$.

Suppose $\perp$ is a polarity of $\mathbf{P}$, then the set of points that are mapped by $\perp$ onto the whole point set of $\mathbf{P}$ forms a subspace of $\mathbf{P}$ called the *radical* of $\perp$ and is denoted by $\mathrm{Rad}(\perp)$. The polarity is called nontrivial if there is a point p such that $p^{\perp}$ is a proper hyperplane of $\mathbf{P}$.

Let $\perp$ be a nontrivial symplectic polarity on $\mathbf{P}$, then the pair $(\mathbf{P}, \perp)$ is called a *symplectic geometry*. For every line l of $\mathbf{P}$ we either have that l is contained in every hyperplane $p^{\perp}$ for all $p \in l$ and l is called a *singular line* or $l \cap p^{\perp} = \{p\}$ for all p on l and we call l *hyperbolic*.

The *geometry of hyperbolic lines* of $(\mathbf{P}, \perp)$ is the partial linear space whose point set is the set of points of $\mathbf{P}$ that are not in the radical of $\perp$ and whose lines are the hyperbolic lines of $\mathbf{P}$ with respect to $\perp$.

If π is a plane of $\mathbf{P}$ then the restriction of $\perp$ to π, where each point $p \in \pi$ is mapped to $p^{\perp} \cap \pi$, is either trivial or has a unique point in its radical. From this we conclude that the geometry of hyperbolic lines of $(\mathbf{P}, \perp)$ has the property that all its planes are symplectic planes as defined in the Introduction.

If the projective space $\mathbf{P}$ contains at least two projective planes, then a well known result of Veblen and Young [17] says that it is Desarguesian and there is a vector space V over some division ring k such that $\mathbf{P} \simeq PV$. Furthermore, if there exists a plane in $\mathbf{P}$ that is disjoint from the radical of the symplectic polarity $\perp$ then there exists a symplectic form f, i.e., a bilinear form f: $V \times V \to k$ with $f(v, v) = 0$ for all $v \in V$, such that for each $v \in V \backslash \{0\}$ we have $\langle v \rangle^{\perp} = \{w \in V \mid f(v, w) = 0\}$. See [3]. In particular in this last situation k is commutative and hence a field.

We finish this section with some comments on symplectic planes. Just as it is possible to reconstruct a projective plane from an affine plane we can also reconstruct a projective plane from a symplectic plane by adding the removed point and lines in the following way.

Let (P, L) be a symplectic plane. Then each point p of P is in a unique maximal coclique of the collinearity graph of (P, L) which we call the *transversal coclique* on p. Suppose ∞ is a new element not in P. Let $\hat{P} = P \cup \{\infty\}$. For each point p define $\langle p, \infty \rangle$ to be the union of the transversal coclique of (P, L) on p together with $\{\infty\}$. Set $\hat{L} = L \cup \{\langle p, \infty \rangle \mid p \in P\}$. Then $(\hat{P}, \hat{L})$ is a projective plane.

3. The Local Structure

Let $\Pi = (P, L)$ be a partial linear space satisfying the hypothesis of Theorem 1.1. In this section we study the local structure of this space, i.e., the structure of the set of points collinear with some fixed one. We start with some general results.

Lemma 3.1. *Suppose π is a plane of Π. Then π is isomorphic to a symplectic plane.*

Proof. By the hypothesis of Theorem 1.1 we see that every plane of Π is contained in a subspace of Π isomorphic to a symplectic plane. However, a symplectic plane is generated by any two lines it contains. ∎

Lemma 3.2. *Let $l \in L$ be a line and $x \in P$ a point not on l. Then x is collinear with no or all but one point of l. The diameter of the collinearity graph of Π is* 2.

Proof. If x is collinear with some point y on l then the subspace of Π generated by l and xy is a symplectic plane and x is collinear with all but one point on l. This proves the first part of the lemma.

Now suppose $p \sim q \sim r \sim s$ is a path in the collinearity graph of Π. Then there is a point t on the line qr that is collinear with both p and s. Hence any minimal path in the collinearity graph has length at most 2. Since Π contains noncollinear points this proves the second part of the lemma. ∎

The above lemma implies that Π is a *copolar space*, see [6]. In fact it is not hard to see that the partial linear spaces considered in this paper are those copolar spaces that are connected partial linear spaces containing a line with at least four points and satisfying Pasch's axiom.

LEMMA 3.3. *Let π be a plane of Π and x a point not in π. Then $\pi \backslash (x^\sim \cap \pi)$ is either π, a line of π, or a transversal coclique of π.*

Proof. Suppose x is collinear with a point y of π. Then on every line of π through y there is a unique point not collinear to x. Fix such a point z_1. Let T be the unique transversal coclique of π on z_1. Every point of T is collinear with y. As π is generated by any three points that are not contained in a line or a transversal coclique we have $T = \pi \backslash \pi \cap x^\sim$ or there is a point z_2 that is not collinear with x and not in T. But then $z_1 z_2$ is in $\pi \backslash \pi \cap x^\sim$ and hence equal to $\pi \backslash \pi \cap x^\sim$. ∎

LEMMA 3.4. *All lines of Π have the same number of points. In particular they all contain at least four points.*

Proof. Suppose there is a line l with s points, for some cardinal number s, and let m be a line intersecting l in a point. Then by examining transversal cocliques in the plane generated by l and m, it is clear that m has also s points. Since Π is connected all lines have s points and $s \geqslant 4$. ∎

LEMMA 3.5. *Let π_1 and π_2 be two planes of Π that meet in at least two noncollinear points. Then $\pi_1 \cap \pi_2$ consists of all or all but one point of a transversal coclique of π_i, $i = 1, 2$.*

Proof. Suppose x and y are two noncollinear points in the intersection of π_1 and π_2. Let $z_i \in \pi_i$, $i = 1, 2$, be two collinear points collinear with x and y. Fix a point u on the line $z_1 z_2$ not collinear to y and let x_1 be the point on $x z_1$ not collinear with z_2. Now let x_2 be the intersection point of $x z_2$ and $x_1 u$ in the symplectic plane generated by $x z_1$ and $x z_2$. We claim that there is at most one line in π_1 on x_1 not meeting π_2.

Suppose l is a line of π_1 containing x_1. Then l meets the line $y z_1$ in a point, r say. If $r = z_1$ or y, then l meets π_2. So assume $r \neq y, z_1$. Since $u \not\sim y$ we have $r \sim u$ and ur meets $y z_2$ in a point s. If $s \sim x_2$, then $s x_2$ and l are in the symplectic plane generated by $u x_1$ and ur, and therefore meet in a point of $\pi_1 \cap \pi_2$. So only if $x_2 \not\sim s$ is it possible that l does not meet π_2. This shows us that all or all but one of the lines of π_1 through x_1 meet π_2. Since $\pi_1 \neq \pi_2$ all points of $\pi_1 \cap \pi_2$ are noncollinear. The lemma follows. ∎

LEMMA 3.6. *Let π be a plane of Π and x a point not in π such that $\pi \backslash x^{\sim}$ is a transversal coclique of π. Then for any two noncollinear points y and z of π that are in $x^{\sim}$ we have that $\langle x, y, z \rangle \cap \pi$ is the transversal coclique of π containing y and z.*

Proof. Suppose y and z are noncollinear points in $\pi \cap x^{\sim}$. Then by the previous lemma $\langle x, y, z \rangle \cap \pi$ contains at least all but one of the points of the transversal coclique T of π containing y and z. Suppose z' is a point of T not in $\langle x, y, z \rangle$. Then $\langle x, y, z' \rangle \cap \pi$ contains also at least all but one point of T. If all lines contain at least five points then T contains at least four points and we can find a point z'' in $\langle x, y, z \rangle \cap \langle x, y, z' \rangle \cap T$ different from y. But then $z' \in \langle x, y, z' \rangle = \langle x, y, z'' \rangle = \langle x, y, z \rangle$, a contradiction. Hence we can assume that all lines of Π contain four points.

Fix points $y = a$, b and c in π such that $x \sim a \sim b \sim c \sim a$, $c \not\sim x \sim b$, and $\pi = \langle a, b, c \rangle$. Then ab contains a unique point d not collinear with x. By the assumptions c and d are noncollinear. Let x_0, r_0 be the points on ax different from a and x. Then $x_0, r_0 \sim d$. Let x_1, respectively r_1, be the intersection point of $x_0 d$, respectively $r_0 d$, with xb. Suppose that y_0 and z_0 are the two points on ac different from a and c. Then $y_0, z_0 \sim d$, and without loss of generality we may assume that $x_0 \sim y_0$ and $r_0 \sim z_0$. Finally set y_1 to be the intersection point of $y_0 d$ and cb, and z_1 the intersection point of $z_0 d$ with cb. Then $z_0 \not\sim x_0$ and $y_0 \not\sim r_0$. Suppose $x_1 \not\sim y_1$. Then $r_1 \not\sim z_1$ and we have that $x_1 \sim y_0$. Furthermore $x_1 \sim c$ and since z_0 is not collinear with x_0 it is also collinear with x_1. Hence x_1 is not collinear with a. By similar arguments (using r and z instead of x and y) we obtain that r_1 is not collinear with a. But that is impossible. Thus we can assume that $x_1 \sim y_1$. Then the lines $x_0 y_0$ and $x_1 y$ are in the plane generated by $x_0 d$ and $y_0 d$ and hence intersect in a point, say f. So $\{x, c, f\} \subseteq \langle x, a, c \rangle \cap \langle x, b, c \rangle$, and as transversal cocliques contain three points we have equality. In particular we have that $\langle x, b, c \rangle \backslash a^{\sim}$ is a transversal coclique.

Now consider the points x, $y = a$, y_1 and z_1 instead of x, a, b, and c. Without loss of generality we can assume that $a \sim z_1$ but $a \not\sim y_1$. By the same arguments as above we obtain that $\langle x, a, y_1 \rangle \cap \langle a, z_1, y_1 \rangle$ is a transversal coclique of $\langle a, y_1, z_1 \rangle = \pi$ containing y and then also z. But then $\langle x, a, y_1 \rangle = \langle x, y, z \rangle$. This finishes the proof of the lemma. ∎

Now fix a point ∞ of Π and consider the geometry $\mathbf{A}_\infty$ whose point set is the set $\infty^{\sim}$. If x and y are two points of $\infty^{\sim}$, then we define the line of $\mathbf{A}_\infty$ through x and y, denoted by l_{xy}, to be the unique transversal coclique of $\langle \infty, x, y \rangle$ containing x and y if x and y are noncollinear in Π, and to be $xy \cap \infty^{\sim}$ otherwise. This definition makes $\mathbf{A}_\infty$ into a linear space.

The following lemma is crucial in the proof of Theorem 1.1.

LEMMA 3.7. *Suppose a, b, and c are three points of $\mathbf{A}_\infty$ that are not on* *a line of* $\mathbf{A}_\infty$. *Then a, b, and c generate a subspace of* $\mathbf{A}_\infty$ *isomorphic to an affine plane.*

Proof. We consider the various possibilities for $\langle a, b, c\rangle$ in Π. First assume that $\langle a, b, c\rangle$ is a symplectic plane. If $\infty \in \langle a, b, c\rangle$ then the lemma follows immediately from the definition of lines in $\mathbf{A}_\infty$ and the structure of symplectic planes.

Suppose $\infty \notin \langle a, b, c\rangle$. Then there are two possibilities for $\langle a, b, c\rangle \backslash \infty^\sim$. It is a transversal coclique or a line of $\langle a, b, c\rangle$.

If $\langle a, b, c\rangle \backslash \infty^\sim$ is a transversal coclique, then it follows by the above lemma that $\langle a, b, c\rangle \cap \infty^\sim$ is closed for lines l_{xy} of $\mathbf{A}_\infty$. Hence in this case $\langle a, b, c\rangle \cap \infty^\sim$ is the point set of a subspace of $\mathbf{A}_\infty$ which is clearly isomorphic to an affine plane.

Now assume that $\langle a, b, c\rangle \backslash \infty^\sim$ is a line of Π. Without loss we can assume that a and b are noncollinear in Π. So $\langle a, b, c\rangle \cap \langle a, b, \infty\rangle$ is all but one point of the transversal coclique in $\langle a, b, \infty\rangle$ which contains a and b. Let d be the unique point on l_{ab} not in $\langle a, b, c\rangle$. To prove the lemma in this particular case, it suffices to show that for all noncollinear points e and f in $\langle a, b, c\rangle \cap \infty^\sim$ the line l_{ef} of $\mathbf{A}_\infty$ contains d. Then a, b, and c generate a subspace of $\mathbf{A}_\infty$ whose point set is $(\langle a, b, c\rangle \cap \infty^\sim) \cup \{d\}$ and which carries the structure of an affine plane.

So let e and f be two noncollinear points in $\langle a, b, c\rangle \cap \infty^\sim$. Assume $e, f \notin l_{ab}$. Then without loss of generality we can assume that there is a point $x \in ae \cap bf \backslash \infty^\sim$. By 3.3 and 3.6 x is not collinear with all points of a line in $\langle \infty, a, b\rangle$. This line has to be ∞d. Let a', respectively b', be collinear points on $a\infty$, respectively $b\infty$, such that $d \in a'b'$. Then $x \sim a', b'$ and xa' meets $e\infty$ in a point, say e'. Since d and x are noncollinear we have $d \sim e'$ and the line de' meets the line xb' in a point f' inside the symplectic plane on xa' and da'. Since x is not collinear with ∞ but b' is, we have that f' is also collinear with ∞. As $\langle x, b, b'\rangle$ is a symplectic plane containing xb and $f'\infty$, these two lines meet in a point, f'' say. But then e is not collinear with f'', for otherwise ef'' would meet $d\infty$ in a point of $\langle x, a, b\rangle$ which is impossible. So $f = f''$ and $\{e, f'', d\}$ is a coclique of the collinearity graph of Π contained in the unique transversal coclique of the symplectic plane $\langle \infty, e', f'\rangle$ containing d and $f'' = f \in l_{ed}$. This shows that $d \in l_{ef}$ and finishes the proof in this case.

Finally we have to consider the case that $\langle a, b, c\rangle$ is not a plane in Π. In this case we can assume that a is not collinear with both b and c in Π. But then there is a point a' on the line $a\infty$ that is collinear with both b and c and we can consider the plane $\langle a', b, c\rangle$ from which we can assume that it does not contain ∞. Let π' be the affine plane in $\mathbf{A}_\infty$ generated by a', b, and c. Consider the set $\mathscr{B}$ composed of the points different from ∞ on the lines ∞q for $q \in \pi'$. This is a subspace of $\mathbf{A}_\infty$ by construction. Each line ∞q

contains a unique point p not collinear with a. Let $\mathscr{C}$ be the set of these points. We claim that $\mathscr{C}$ is also a subspace of $\mathbf{A}_\infty$. Let $x, y \in \mathscr{C}$. First assume that $a = x$. Then l_{xy} is a coclique containing x and so is in $\mathscr{C}$. Next suppose $x \neq a \neq y$., Then $\langle \infty, x, y \rangle \backslash a^\sim$ contains x, y and so l_{xy} by 3.3. Again $l_{xy} \subseteq \mathscr{B} \cap \mathscr{C}$. Thus $\mathscr{C}$ is a subspace of $\mathbf{A}_\infty$ which by projection from ∞ is isomorphic to the affine plane π'. This proves the lemma. ▮

The above lemma has the following important consequence.

PROPOSITION 3.8. *Suppose Π contains a line with at least five points. Then for each point ∞ of Π the space $\mathbf{A}_\infty$ is isomorphic to an affine space.*

Proof. This follows by the above lemma and Buekenhout [2]. ▮

The above result also holds in the case that all lines of Π contain four points. However, in that case Buekenhout's result cannot be applied and some more work is required which is done in Section 5.

The idea of considering the local structure at a point ∞ also appears in the work of Lefèvre–Percy [12] and of Thas and DeClerck [14]. Versions of Lemma 3.5 and 3.6 can be found in [14].

4. THE EMBEDDING IS A SYMPLECTIC GEOMETRY

Let $\Pi = (P, L)$ be a partial linear space as in Theorem 1.1. In this section we show how one can embed Π in a symplectic geometry provided there is a point $\infty \in P$ such that the linear space $\mathbf{A}_\infty$ is affine.

So assume $\infty \in P$ such that $\mathbf{A}_\infty$ is isomorphic to an affine space.

Let $\mathbf{P}_\infty$ be a projective space containing $\mathbf{A}_\infty$ as the complement of some hyperplane. We construct an embedding of Π into $\mathbf{P}_\infty$.

Let x be a point in Π not collinear to ∞. Every line l of L on x that meets $\infty^\sim$ nontrivially determines a unique line $l \cap \infty^\sim$ in $\mathbf{A}_\infty$. Since Π is connected and the collinearity graph of Π has diameter at most 2 there exists at least one line through x that meets $\infty^\sim$ nontrivially. Furthermore, two lines on x meeting $\infty^\sim$ determine parallel lines in $\mathbf{A}_\infty$. Thus to every point x of P not collinear with ∞ we can adjoin the parallel class of lines of $\mathbf{A}_\infty$ that contains the lines $l \cap \infty^\sim$ where l is a line of L through x meeting $\infty^\sim$ nontrivially. This parallel class is denoted by $[x]$.

Now consider the map ϕ from P into the point set of $\mathbf{P}_\infty$ defined by

$$\phi(x) = x \qquad \text{if} \quad x \in \infty^\sim;$$

$$\phi(x) = [x] \qquad \text{otherwise.}$$

LEMMA 4.1. *The map ϕ is injective.*

Proof. Let x and y be two points of Π. If x or y is collinear to ∞ then clearly $\phi(x) \neq \phi(y)$. So assume that both x and y are not in $\infty^{\sim}$. To prove the lemma it suffices to find lines through x and y that intersect in a point of $\infty^{\sim}$. If x or y is equal to ∞ this is straightforward. Thus let us assume that they are both different from ∞. Then as the diameter of the collinearity graph of Π is 2 there are points z_1 and z_2 with $\infty \sim z_1 \sim x$ and $\infty \sim z_2 \sim y$. Then $\langle z_1, z_2, \infty \rangle$ is a symplectic plane or a line and it follows by 3.3 that there is a point z in this subspace that is collinear to ∞ and both x and y. So xz and yz are the lines we are looking for. ∎

LEMMA 4.2. *ϕ maps lines of Π onto lines of $\mathbf{P}_\infty$.*

Proof. Let l be a line in L and consider $\phi(l) = \{\phi(x) \mid x \in l\}$. If l meets $\infty^{\sim}$ then $\phi(l)$ is the point set of a line in $\mathbf{P}_\infty$. So suppose that l does not contain a point in $\infty^{\sim}$. Then let m be a line meeting l and containing some point collinear to ∞. The symplectic plane $\langle l, m \rangle$ meets $\infty^{\sim}$ in $\langle l, m \rangle \backslash l$, and for each point x of l the parallel class $[x]$ contains a line in the affine plane π of $\mathbf{A}_\infty$ generated by $\langle l, m \rangle \cap \infty^{\sim}$. Furthermore, every line of π is in some parallel class $[x]$ where $x \in l$. So again $\phi(l)$ is the full point set of a line in $\mathbf{P}_\infty$. ∎

The above lemmas show that ϕ induces an embedding of Π into $\mathbf{P}_\infty$. In the remainder of this section we show that there exists a symplectic polarity $\perp$ on $\mathbf{P}_\infty$ such that the image of Π under ϕ is the geometry of hyperbolic lines in $(\mathbf{P}_\infty, \perp)$.

The map $\perp$ is defined as follows. Let p be a point of $\mathbf{P}_\infty$, then $p^\perp$ is all of $\mathbf{P}_\infty$ if $p \notin \phi(P)$ and equals the complement of $\{\phi(y) \mid y \sim x\}$ in $\mathbf{P}_\infty$ if $p = \phi(x)$ for some $x \in P$.

LEMMA 4.3. *Let $R = \mathbf{P}_\infty \backslash \phi(P)$. Then R is a subspace of $\mathbf{P}_\infty$.*

Proof. Note first that a point of $\mathbf{P}_\infty$ is an element of R if and only if it is a parallel class of $\mathbf{A}_\infty$ all of whose elements l are transversal cocliques in a symplectic plane of Π containing ∞. These lines l of $\mathbf{A}_\infty$ are characterized by the property that for each point x that is not collinear with ∞ one has that $x^{\sim} \cap l$ equals either l or $\varnothing$.

Now let π be an affine plane in $\mathbf{A}_\infty$ containing two intersecting lines whose parallel classes are in R. Then each line in π is a transversal coclique of some symplectic plane on ∞ and for every point x not collinear with ∞ we either have $x^{\sim} \cap \pi = \pi$ or $\varnothing$. Hence all lines in π are in a parallel class which is in R. This proves that R is a subspace. ∎

LEMMA 4.4. *The map $\perp$ is a symplectic polarity on $\mathbf{P}_\infty$ with radical R.*

Proof. Let p be a point of $\mathbf{P}_\infty$. First we show that $p^\perp$ is a hyperplane of $\mathbf{P}_\infty$. Without loss we can assume that $p = \phi(x)$ for some $x \in P$. Let l be a line of $\mathbf{P}_\infty$. If l contains at least two points of R then by the above lemma it is contained in $R \subseteq p^\perp$.

Thus we may assume that there are at least two points y and z in P with $\phi(y)$ and $\phi(z)$ in l. Suppose that l contains a point not in $p^\perp$. This point is not in R, so without loss we can assume that it is the point $\phi(y)$. Then $x \sim y$. If also $z \sim y$ then the point $\phi(u)$ is the unique point of l in $p^\perp$ where u is the unique point on l not collinear with x. If z is not collinear with y then let u be a point collinear in Π both y and z and to ∞. Such a point exists as we saw before. For every point v of the transversal coclique on y and z in the symplectic plane generated by u, y, and z we have that $\phi(v)$ is on the line through $\phi(y)$ and $\phi(z)$ which is l. Without loss we can now assume that x and z are collinear so that $\langle x, y, z \rangle$ is a symplectic plane in Π. This symplectic plane is embedded by ϕ in the projective subspace of $\mathbf{P}_\infty$ generated by p and l. Within this subspace one easily checks that l contains a unique point in $p^\perp$. This shows us that $p^\perp$ is a hyperplane of $\mathbf{P}_\infty$.

Now suppose that q is a point in $p^\perp$. If p or q is in R then $p \in q^\perp$. Thus assume that both points are outside R. Then there are x and y in P such that $p = \phi(x)$ and $q = \phi(y)$. Since $q \in p^\perp$ we have $x \not\sim y$ but then also $y \not\sim x$ and $p \in q^\perp$. Hence $\perp$ is a polarity. The polarity $\perp$ is nontrivial since Π contains collinear points, its radical is clearly R. That it is a symplectic polarity follows directly from the definition of $\perp$. ∎

Proposition 4.5. *Let Π be a partial linear space satisfying the conditions of Theorem* 1.1. *Suppose furthermore that there is a point ∞ in Π such that $\mathbf{A}_\infty$ is an affine space. Then Π is isomorphic to the geometry of hyperbolic lines in some symplectic geometry.*

Proof. The map ϕ induces an isomorphism between Π and the geometry of hyperbolic lines in the symplectic geometry $(\mathbf{P}_\infty, \perp)$. ∎

5. Lines with 4 Points

In this section we are concerned with proving the analogue of 3.8 in the case that all lines in L contain four points.

So assume that $\Pi = (P, L)$ is a partial linear space satisfying the conditions of Theorem 1.1 and all lines in L contain four points. As in Section 3 fix a point ∞ and consider the linear space $\mathbf{A}_\infty$. We show that $\mathbf{A}_\infty$ is an affine space. The crucial lemma of this section is the following.

Lemma 5.1. *Let a, b, and c be three points in $\infty^\sim$ that generate an affine plane in $\mathbf{A}_\infty$. Suppose that ∞ is not contained in the subspace of Π generated* *by a, b, and c. Then $\langle \infty, a, b, c \rangle$ is isomorphic to the geometry of hyperbolic lines in a three-dimensional symplectic geometry.*

Proof. Let A be the set of $9.3 = 27$ points on the lines through ∞ meeting the affine plane π in $\mathbf{A}_\infty$ generated by a, b, and c and different from ∞. If x and y are two points in A then there are points x' and y' in π with $\langle \infty, x, y \rangle \subseteq \langle \infty, x', y' \rangle$. But then l_{xy} is contained in A. So A is the point set of a subspace of $\mathbf{A}_\infty$. In fact it is isomorphic to an affine 3-space.

Let $\hat{A}$ be the union of all lines in L that meet A in at least two points. We show that $\hat{A}$ is a subspace of Π. Let x and y be two collinear points in $\hat{A}$. We have to show that $xy \subseteq \hat{A}$. If both x and y are in A this is clear. So assume that $x \notin A$. Then there is a line $l \in L$ meeting A in at least two points and containing x. If $y \in A$ and on l there is nothing to prove. If y is not on l then xy is contained in the subspace of Π generated by y and l. Since x and y are collinear this subspace is isomorphic to a symplectic plane which meets A in an affine plane of $\mathbf{A}_\infty$. In particular, xy meets A in $xy \backslash \{x\}$. So xy is contained in $\hat{A}$.

Finally suppose that $y \notin A$. Then there is a line $m \in L$ that contains y and meets A in at least two points. If l meets m then $xy \subseteq \langle l, m \rangle$ which is a symplectic plane meeting A in an affine plane of A_∞. As above it is now easy to check that $xy \subseteq \hat{A}$.

So assume l does not meet m. As we can replace l (respectively, m) by any line on x (respectively, y) in $\langle \infty, l \rangle$ (respectively, $\langle \infty, m \rangle$) we may assume that $l \cap A$ and $m \cap A$ are parallel lines in $\mathbf{A}_\infty$ and that there is a point z on $m \cap A$ collinear in Π with some point on $l \backslash \{x\}$. But then $l, m \subseteq \langle z, l \rangle$ which meets A in an affine plane. Hence $x, y \in l \cap m$ and $x = y$ against our assumptions.

We can conclude that $\hat{A}$ is a subspace of Π which is locally A in the point ∞. But now the result follows from Proposition 4.5. ∎

Suppose l is a line of $\mathbf{A}_\infty$. Then either l is a transversal coclique in some symplectic plane on ∞ or there is a line $\hat{l}$ in L with $l = \hat{l} \cap \infty^{\sim}$. In this last case we call the unique point on $\hat{l}$ which is not collinear with ∞ the *point at infinity* of l.

If l is a transversal coclique, then by $T(l)$ we denote the set of all transversal cocliques (in some symplectic plane) meeting l in at least two points. In this case we define $\hat{l} = \bigcup_{t \in T(l)} t$. Note, in this case $\hat{l}$ is defined independently of ∞.

LEMMA 5.2. *Let l be a transversal coclique of Π. Then either $\hat{l} = l$ or there is a unique point in $\hat{l} \backslash l$. Moreover, for every $t \in T(l)$ we have $\hat{t} = \hat{l}$.*

Proof. If $\hat{l} = l$ there is nothing to prove. Thus assume $\hat{l} \neq l = \{x, y, z\}$. Then there are symplectic planes π_1 and π_2 in Π such that $l \subseteq \pi_1$ and

$l \not\subseteq \pi_2 \supseteq \{x, y\}$ (up to permutation of $\{x, y, z\}$). By Lemma 5.1 we have that $\langle \pi_1, \pi_2 \rangle$ is isomorphic to the geometry of hyperbolic lines in a three-dimensional symplectic space. Since $l \not\subseteq \pi_1$ we see that the radical of the symplectic polarity defining the symplectic space is empty. But then we can find a point u in $\hat{l} \backslash l$ inside this subspace. Furthermore for every point $v \in \{x, y, z, u\}$ there is a symplectic plane $\pi_v \subseteq \langle \pi_1, \pi_2 \rangle$ that contains $\{x, y, z, u\} \backslash \{v\}$. Now suppose π is a symplectic plane in Π containing x and y. Fix a point p in π collinear with both x and y. Then p is collinear with at least one of the points z or u. Then there is a symplectic plane π_x that contains $\{u, y, z\}$ as a transversal coclique. So Lemma 3.3 applies to p and the plane π. Without loss we can assume that p is collinear with z. So by 3.6 we have $\pi \cap \pi_1 = \{x, y, z\} \subseteq l \cup \{u\}$, and it follows that $\hat{l} = \{u, x, y, z\}$.

For any $t_1, t_2 \in T(l)$ we have $|t_i \cap \hat{l}| = 3$, $i = 1, 2$. So $|t_1 \cap t_2| \geqslant 2$. But that implies that $T(l) = T(t)$ for any $t \in T(l)$, and we have proved the lemma. ∎

Let l be a line of $\mathbf{A}_\infty$ which is a transversal coclique. If $\hat{l} \neq l$ then the unique point in $\hat{l} \backslash l$ is called the *point at infinity* of l. If $\hat{l} = l$ we say that l has no point at infinity.

LEMMA 5.3. *Let l be a line in $\mathbf{A}_\infty$ which is a transversal coclique. Suppose x is the point at infinity of l. Let $y \in P$ be collinear with x. Then there is a symplectic plane containing xy and meeting l in at least two points.*

Proof. Let π be some symplectic plane meeting l in at least two points and containing x. Let l' be the transversal coclique of π containing x. By Lemma 3.3 y is collinear with at least two points of l', x and z say. Then $\langle x, y, z \rangle$ is a symplectic plane containing xy and meeting l' in at least two points and thus $\hat{l}' = \hat{l}$ in three points, one being x. This proves the lemma. ∎

Suppose l and m are two lines in $\mathbf{A}_\infty$. Then we define $l \| m$ if and only if there is an affine plane in $\mathbf{A}_\infty$ such that l and m are parallel in that plane. To show that $\mathbf{A}_\infty$ is an affine space we have to show that $\|$ is an equivalence relation. The relation is certainly reflexive and symmetric, so we only have to consider whether it is a transitive relation.

LEMMA 5.4. *Let l be a line of $\mathbf{A}_\infty$ which is the intersection of a line $\hat{l}$ of Π with $\infty^\sim$ and suppose m is another line of $\mathbf{A}_\infty$. Then $l \| m$ if and only if m and l have the same point at infinity.*

Proof. Suppose $l \| m$. Then the subspace $\langle \infty, l, m \rangle$ of Π equals $\langle \infty, l, y \rangle$ for every point y on m. But then one can check in $\langle \infty, l, y \rangle$, which is known by 5.1 and contains the point at infinity of l, that this point is also the point at infinity of m.

Now suppose x is the point at infinity of both m and l. Then by the above lemma there is a symplectic plane π on l that meets m in at least two points. (If m is not a transversal coclique then $\pi = \langle l, m \rangle$.) But then we can check inside $\langle \infty, \pi \rangle$, which is known by 5.1 and contains l, m, and x, that $l \| m$. ∎

LEMMA 5.5. *Let l, m, and n be lines of $\mathbf{A}_\infty$ that are transversal cocliques. If $l \| m \| n$, then $l \| n$.*

Proof. Suppose $l \| m \| n$ and $l \neq m$. If n, m are contained in the symplectic plane of Π generated by ∞ and l, then the lemma is clear. So suppose $m \subseteq \langle \infty, l \rangle \not\supseteq n$. Then the 27 points in $\infty^\sim$ on the 9 lines through ∞ meeting the affine plane of $\mathbf{A}_\infty$ on m and n form an affine 3-space containing l. Inside this space we already have that $l \| n$.

The above shows that we may replace any line k in $\{l, m, n\}$ by a line k' parallel to k in $\infty^\sim \cap \langle \infty, k \rangle$. Suppose that both m and n are not in $\langle \infty, l \rangle$. The above observation implies that we can assume that there are symplectic planes π_1, π_2, and π_3 with $l, m \subseteq \pi_1$, $m, n \subseteq \pi_2$, and $x, l \subseteq \pi_3$ for some point x in n. But then n and π_3 are contained in $\langle \pi_1, \pi_2 \rangle$ which is known by 5.1 Inside this space one checks that $n \subseteq \pi_3$ from which it follows that $l \| n$. ∎

LEMMA 5.6. *Let l and m be two lines in $\mathbf{A}_\infty$ that are transversal cocliques. Suppose x is the point at infinity of l. Then $l \| m$ if and only if x is also the point at infinity of m.*

Proof. Suppose $l \| m$. First assume that $m \subseteq \langle \infty, l \rangle$. Let y be a point in P collinear with ∞ and x. Then by Lemma 5.4 there is a symplectic plane π of Π containing xy and meeting l in at least two points. By 5.1 the space $\langle \infty, \pi \rangle$ is known and contains l, xy, and m. Inside this subspace we can check that x is the point at infinity of m.

By the above and Lemma 5.5 we may replace l by some line l' in $\mathbf{A}_\infty$ that is in $\langle \infty, l \rangle$ and $l \| l'$. So without loss we can assume that there is a symplectic plane π' of Π that contains both l and m. But then consider the subspace $\langle \pi, \pi' \rangle$ of Π. Inside this space we can find a symplectic plane containing x and at least two points of m. But then x is in $\hat{m}$ and thus the point at infinity of m.

Now suppose x is the point at infinity of m. Fix a point $y \in m$. In the subspace of $\mathbf{A}_\infty$ generated by l and y there is a line m' of $\mathbf{A}_\infty$ with $m' \| l$. By the above $x, y \in \hat{m}'$, and by Lemma 5.2, we have $x \in \hat{m}' = \hat{m}$ proving the lemma. ∎

LEMMA 5.7. *The relation $\|$ is transitive.*

Proof. Suppose $l \| m \| n$ for some lines l, m, and n of $\mathbf{A}_\infty$. If all the three lines are transversal cocliques the lemma follows by 5.5. Thus suppose at least one of them is not a transversal coclique. Then by 5.4 and 5.6 they all have the same point at infinity and $l \| n$. ▮

PROPOSITION 5.8. *The space $\mathbf{A}_\infty$ is isomorphic to an affine space.*

Proof. This follows by the above lemma. ▮

Theorem 1.1 follows now from the above Proposition 5.8, Proposition 3.8, Proposition 4.5, and the classical result of Veblen and Young [17].

6. SYMPLECTIC TRANSVECTION GROUPS

In this section we discuss some properties of symplectic transvection groups needed in the proof of Theorem 1.2 and 1.3. Most of these results are well known and can be found in, for example, [3, 11, 15].

Let k be a division ring and V a left k-vector space and V^* its dual space. Then for each vector $v \in V \setminus \{0\}$ and $\phi \in V^* \setminus \{0\}$ with $v\phi = 0$ we define a linear map $t(\phi, v)$: $V \to V$ by

$$wt(\phi, v) = w + (w\phi)v \qquad \text{for all} \quad w \in V.$$

This map $t = t(\phi, v)$ is called a *transvection* on V. The space $C_V(t) = \{w \in V | w\phi = 0\}$ is called the *axis* of t and the one-dimensional space $\langle v \rangle = [V, t] = \{w(t-1) | w \in V\}$ is the *center* of t. In fact any nontrivial linear transformation t with $[V, t]$ being one-dimensional and in $C_V(t)$ is a transvection.

Suppose v and ϕ are nonzero elements of V and V^*, respectively. A *transvection subgroup* $T(\phi, v)$ is the subgroup of the linear group on V generated by all the transvections $t(\psi, w)$ where $\psi \in \langle \phi \rangle \setminus \{0\}$ and $w \in \langle v \rangle \setminus \{0\}$. Clearly $T(\phi, v)$ is isomorphic to the additive group of k. It contains all the transvections with center $\langle v \rangle$ and axis $\{w \in V | w\phi = 0\}$. For subsets W of V and R of V^* we denote by $T(R, W)$ the group generated by all the transvection subgroups $T(\psi, w)$ where ψ and w are nonzero elements in R and W, respectively, with $w\psi = 0$.

LEMMA 6.1. *Let $T(\phi, v)$ and $T(\psi, w)$ be two transvection subgroups of $GL(V)$, $v, w \in V \setminus \{0\}$, and $\phi, \psi \in V^* \setminus \{0\}$. Then the following hold*:

(i) *if $w\phi = v\psi = 0$ then $[T(\phi, v), T(\psi, w)] = 1$;*

(ii) *if $w\phi \neq 0 \neq v\psi$ then $T(\phi, v)$ and $T(\psi, w)$ are conjugate full unipotent subgroups of $\langle T(\phi, v), T(\psi, w) \rangle$ which is isomorphic to $SL(2, k)$.*

The group $T(\phi, v)$ *is regular on the set of transvection subgroups in* $\langle T(\phi, v), T(\psi, w)\rangle$ *different from* $T(\phi, v)$.

Proof. See [3, Lemma 3.2]. ▮

Let $V = U \oplus W$ where U is a two-dimensional subspace of V. Suppose $R = \mathrm{Ann}_{V^*}(W) = \{\phi \in V^* \mid w\phi = 0 \text{ for all } w \in W\}$.

LEMMA 6.2. *The group* $T(R, V\backslash W)$ *is isomorphic to a split extension of a direct sum of* dim W *copies of* U^* *by* $SL(U)$.

Proof. Clearly $T(R, V\backslash W)$ is a split extension of the stabilizer M in $GL(V)$ of the chain $0 \leqslant W \leqslant V$ by $SL(U)$. As in [10, 3.13] and [11, 3.6] we see that M is isomorphic to a direct sum of dim W copies of U^* on which $SL(U)$ acts naturally. ▮

LEMMA 6.3. *The transvection subgroups of* $T(R, V\backslash W)$ *form a conjugacy class of k-transvection subgroups of symplectic type in* $T(R, V\backslash W)$.

Proof. Let $v \in V\backslash W$. Then $R_v = \{\phi \in R \mid v\phi = 0\}$ is a unique one-dimensional subspace of V^*. Hence for each $v \in V\backslash W$ the subgroup $T(R_v, v)$ is the unique transvection subgroup with center $\langle v\rangle$ which is contained in $T(R, V\backslash W)$. Now for any vector w in $V\backslash W$ we either have $w \in \langle v, W\rangle$ and then $w\phi = 0$ for all $\phi \in R_v$ or $w \notin \langle v, W\rangle$, and there exists an element $\phi \in R_v$ with $w\phi \neq 0$. So two transvection subgroups either commute or are full unipotent subgroups in the group they generate which is isomorphic to $SL(2, k)$. Since W has codimension 2 in V we find that the transvection subgroups form a conjugacy class in $T(R, V\backslash W)$. That this class is a class of k-transvection subgroups of symplectic type follows now by Lemma 6.2 and the fact that for any three-dimensional subspace V_1 of V which intersects W in a one-dimensional subspace we have that $T(R, V_1\backslash(W \cap V_1))$ is isomorphic to $T(R_1, V_1\backslash W_1)$ in $GL(V_1)$, where W_1 is the one-dimensional subspace $V_1 \cap W$ of V_1 and $R_1 = \mathrm{Ann}_{V_1^*}(V_1 \cap W)$. ▮

Suppose there exists a nontrivial symplectic form f on V. (So k is commutative.) Then there exists a canonical map $v \mapsto \phi_v$ of V into V^* defined by $w\phi_v = f(v, w)$ for all $w \in V$. The *radical* of f denoted by $\mathrm{Rad}(f)$ is the kernel of this map. We call the space (V, f) a *symplectic space*. It is called *nondegenerate* if $\mathrm{Rad}(f) = 0$ and *degenerate* otherwise.

The symplectic group $\mathrm{Sp}(V, f)$ consists of all linear transformations $g \in GL(V)$ leaving f invariant. If $t = t(\phi, v)$ is a transvection in $\mathrm{Sp}(V, f)$ then an easy calculation shows that either $v \in \mathrm{Rad}(f) \subseteq C_V(t)$ or $\phi \in \langle \phi_v\rangle$. So for each vector v in $V\backslash\mathrm{Rad}(f)$ the only transvections in $\mathrm{Sp}(V, f)$ with center $\langle v\rangle$ are the transvections in $T(\phi_v, v)$. These transvection subgroups are called the *symplectic transvection subgroups* (with respect to f).

By $\mathrm{Sp}^+(V, f)$ we denote the subgroup of $\mathrm{Sp}(V, f)$ fixing $\mathrm{Rad}(f)$ pointwise. Its subgroups generated by the symplectic transvection subgroups is denoted by $T\mathrm{Sp}(V, f)$. As any element g in $T\mathrm{Sp}(V, f)$ is a finite product of transvections, the centralizer in V of g has finite codimension. So $[V, g]$ is finite dimensional. Thus $T\mathrm{Sp}(V, f)$ is also a subgroup of the finitary groups $F\mathrm{Sp}(V, f)$ and $F\mathrm{Sp}^+(V, f)$ which consist of all elements $g \in \mathrm{Sp}(V, f)$, respectively $\mathrm{Sp}^+(V, f)$, with $[V, g]$ being finite dimensional.

LEMMA 6.4. *Suppose f is a nontrivial symplectic form on V.*

(i) *If f is nondegenerate, then $T\mathrm{Sp}(V, f) = F\mathrm{Sp}(V, f)$;*

(ii) *if f is degenerate and W is a complement to* $\mathrm{Rad}(f)$ *in V, then $T\mathrm{Sp}(V, f)$ is isomorphic to a split extension of a direct sum of copies of W by $F\mathrm{Sp}(W, f|_W)$;*

(iii) *$T\mathrm{Sp}(V, f)$ has a nontrivial center only if V is finite dimensional and f is nondegenerate. In this case the center has order at most* 2.

Proof. The proof is similar to 3.13 of [10] and 3.6 of [11] and therefore is left to the reader. ∎

Let G be $T\mathrm{Sp}(V, f)$. The images of transvections and transvection subgroups in $G/Z(G)$ are also called transvections, respectively transvection subgroups. We close this section with some properties of the class of symplectic transvection subgroups.

LEMMA 6.5. *Suppose f is a nontrivial symplectic form on V. Let G be the group $T\mathrm{Sp}(V, f)$ or $T\mathrm{Sp}(V,f)/Z((T\mathrm{Sp}(V, f))$ and Σ the set of all symplectic transvection subgroups of G.*

(i) *Σ is a conjugacy class of k-transvection subgroups of symplectic type in G.*

Moreover, if (V, f) is nondegenerate, then

(ii) *if $A, B \in \Sigma$ with $[A, B] \neq 1$, then $G = \langle C_\Sigma(A), B\rangle$;*

(iii) *if A, B, and C are in Σ and $[A, B] \neq 1 \neq [A, C]$, then $G = \langle C_\Sigma(C), C_\Sigma(A) \cap C_\Sigma(B)\rangle$ or $C \in \langle A, B\rangle \cap \Sigma$.*

Proof. Let v and w be two vectors in $V \backslash \mathrm{Rad}(f)$. Then either $f(v, w) = 0$ and the two transvection subgroups $T(\phi_v, v)$ and $T(\phi_w, w)$ commute or $f(v, w) \neq 0$ and these two groups are conjugate full unipotent subgroups of $\langle T(\phi_v, v), T(\phi_w, w)\rangle$ which is isomorphic to $SL(2, k)$. So to prove that Σ is a conjugacy class we can assume that $f(v, w) = 0$ and it suffices to find a vector u with $f(v, u) \neq 0 \neq f(w, u)$. As v and w are not in the radical of f there is a vector u_x with $f(x, u_x) \neq 0$, with $x = v, w$. If $f(w, u_v)$ or $f(v, u_w)$ is nonzero then we are done. So suppose they are both 0. But then $f(v, u_v + u_w) \neq 0 \neq f(w, u_v + u_w)$.

If u, v, and w are three vectors spanning a three-dimensional subspace of V, with $f(u, v) \neq 0 \neq f(u, w)$, then the form f restricted to $V_1 = \langle u, v, w \rangle$ has a one-dimensional radical $R = \mathrm{Rad}(f|_{V_1})$. The group $\langle T(u, \phi_u), T(v, \phi_v), T(w, \phi_w) \rangle$ is isomorphic to $T(\mathrm{Ann}_{V_1^*}(R), V_1 \backslash R)$. So by Lemma 6.2 Σ is a class of k-transvection subgroups of symplectic type in any central quotient of $T\,\mathrm{Sp}(V, f)$.

To prove (ii) we can assume that $G = T\,\mathrm{Sp}(V, f)$. Suppose f is nondegenerate and let A and B be two noncommuting elements in Σ with centers $\langle v \rangle$ and $\langle w \rangle$, respectively. Let C be an element in Σ with center $\langle u \rangle$ that does not commute with A. Let x be a nonzero vector in $\langle u, w \rangle$ with $f(v, x) = 0$. Then $C \subseteq \langle B, T(\phi_x, x) \rangle$. Hence Σ and thus G is contained in $\langle C_\Sigma(A), B \rangle$. This proves (ii).

Finally, (iii) follows by (ii) and the observation that $\{x \in V \mid f(x, y) = 0$ for all y with $f(v, y) = f(w, y) = 0\}$ equals $\langle v, w \rangle$. ∎

7. Transvection Subgroups of Symplectic Type

Let G be a centerfree group and k a division ring with at least three elements. Suppose Σ is a conjugacy class of k-transvection subgroups of symplectic type in G. In this section we prove Theorem 1.2.

We start with some properties of $(P)SL(2, k)$.

Lemma 7.1. *Let A and B be two noncommuting elements of Σ. Suppose $X = \langle A, B \rangle$. Then the following hold.*

(i) *The full unipotent subgroups in X are conjugate;*

(ii) *A acts (by conjugation) regularly on the set of full unipotent subgroups of X different from A;*

(iii) *X is generated by A and any element $x \in X \backslash N_X(A)$;*

(iv) *if C and D are elements of Σ with $X \leqslant \langle C, D \rangle$ then $X = \langle C, D \rangle$.*

Proof. (i) and (ii) are well known, (iii) and (iv) follow directly from (i) and (ii). See also [15]. ∎

Let $L(\Sigma)$ be the set $\{\langle A, B \rangle \cap \Sigma \mid A, B \in \Sigma$ with $[A, B] \neq 1\}$. Then the above lemma has the following

Corollary 7.2. *$(\Sigma, L(\Sigma))$ is a connected partial linear space in which every pair of intersecting lines generates a plane isomorphic to a symplectic plane. Moreover, all lines contain at least four points.*

Proof. By the above lemma every pair of noncommuting elements of Σ

is in a unique element of $L(\Sigma)$. Thus $(\Sigma, L(\Sigma))$ is a partial linear space. This space is connected since Σ is a conjugacy class in $G = \langle \Sigma \rangle$. Every line of this space contains $|k| + 1$ points and any two intersecting lines generate a subgroup which modulo its center is isomorphic to a split extension of the dual of the natural $SL(2, k)$-module by $SL(2, k)$. Thus, up to a center, it is isomorphic to $T\ (R, V \backslash U)$ for some three-dimensional k-vector space V with one-dimensional subspace U and $R = \mathrm{Ann}_{V^*}(U)$. But then the subspace of $(\Sigma, L(\Sigma))$ generated by two intersecting lines is isomorphic to the space of one-dimensional subspace of V different from U and two-dimensional subspaces intersecting U trivially, and thus to a symplectic plane. ∎

If $(\Sigma, L(\Sigma))$ contains no or one plane, then there is nothing to prove. Thus we can assume that there are at least two planes in this space. This means we can apply Theorem 1.1. Hence there exists a vector space V over some division ring k' and a symplectic polarity $\perp$ on PV, such that $(\Sigma, L(\Sigma))$ is isomorphic to the geometry of hyperbolic lines in $(PV, \perp)$. Identify these two spaces. Then there is a homomorphism of G in the automorphism group of $(PV, \perp)$. The kernel of this homomorphism normalizes every element of Σ and thus centralizes it. Hence the kernel is contained in the center of G which is trivial, cf. [8]. Let A be an element of Σ. Then every $a \in A$ fixes A and $A^{\perp}$ pointwise and induces an elation on PV.

Thus we can lift the group G into $GL(V)$ so that the elements of Σ are lifted into transvection subgroups. Since the group $A \in \Sigma$ is transitive on the points different from A on every line through A not contained in $A^{\perp}$, it follows by 6.1 that A lifts onto the full transvection subgroup of $GL(V)$ with center A and axis $A^{\perp}$ (seen as subspaces of V).

Now we can distinguish two situations. First assume that the radical of $\perp$ has codimension 2. Let $U \leqslant V$ be this radical. Then clearly G is mapped by the above homomorphism onto $T(R, V \backslash U)$, where $R = \mathrm{Ann}_{V^*}(U)$.

Next assume that the radical has codimension at least 3. Then there exists a symplectic form f on V that represents $\perp$. But then A lifts onto the full symplectic transvection group of $\mathrm{Sp}^+(V, f)$ with center A. So the above homomorphism maps G onto $T\mathrm{Sp}(V, f)$ modulo its center. In both cases the elements of Σ are mapped onto the class of symplectic transvection groups of the apropriate group. But then $k' \simeq k$ and we have finished the proof of Theorem 1.2.

8. Symplectic Modules

This last section is devoted to the proof of Theorem 1.3. So let G be the group $F\mathrm{Sp}(V, f)$ for some nondegenerate symplectic space (V, f) over the field k of dimension at least 4. Let Σ be the conjugacy class of symplectic transvection subgroups in G. Suppose M is a nontrivial $\mathbb{Z}G$-module satisfying the hypothesis of 1.3.

LEMMA 8.1. *Suppose A and B are two noncommuting elements of Σ. Then the following hold.*

(i) $M = C_M(A) + C_M(B)$;

(ii) *if $C \in \Sigma$ such that $C \notin \Sigma \cap \langle A, B\rangle$ and C does not commute with A, then $M = C_M(C) + C_M(\langle A, B\rangle)$.*

Proof. By the hypothesis of the theorem we have $C_M(A) \supseteq [M, C_\Sigma(A)]$ for all $A \in \Sigma$. But then the lemma follows from Lemma 6.5. ∎

Now consider the group $\hat{G}$ which is the split extension of M and G. Let $\hat{\Sigma}$ be the conjugacy class of $\hat{G}$ containing Σ.

PROPOSITION 8.2. *$\hat{\Sigma}$ is a conjugacy class of k-transvection subgroups of $\hat{G}$ of symplectic type.*

Proof. First we note that $[M, A] \leqslant C_M(B)$ for all $B \in \Sigma$ commuting with A. So, since $[A, A] = 1$, any element of $\hat{\Sigma}$ is of the form (M_0, A) where $M_0 \leqslant [M, A]$ and $A \in \Sigma$. Since $[M, G] = M$ we find that $\hat{\Sigma}$ is just the set of all elements in $\hat{G}$ of that form and generates $\hat{G}$. Furthermore, if $A, A' \in \Sigma$ and M_0, M_0' are contained in $[M, A]$, respectively $[M, A']$, with (M_0, A) and (M_0', A') being elements of $\hat{\Sigma}$, then (M_0, A) and (M_0', A') commute if and only if A and A' commute.

Let (m, A) and (n, B) be two noncommuting elements of $\hat{\Sigma}$. Then by (i) of the above lemma we can conjugate them simultaneously to $(0, A)$ and $(0, B)$. So they are full unipotent subgroups of the subgroup of $\hat{G}$ they generate, which is isomorphic to $\langle A, B\rangle$. This implies that $\hat{\Sigma}$ is a conjugacy class of k-transvection subgroups in $\hat{G}$. It remains to show that this class is of symplectic type. Thus let (m, A), (n, B), and (l, C) be three elements of $\hat{\Sigma}$ such that $[A, B] \neq 1 \neq [A, C]$. If C is not contained in $\langle A, B\rangle$ then it follows from 6.5(iii) that we can conjugate the above elements simultaneously to $(0, A)$, $(0, B)$, and $(0, C)$, respectively. In that case these three elements generate a subgroup of $\hat{G}$ which modulo its center is isomorphic to $k^2 \colon SL(2, k)$.

So suppose that C is contained in $\langle A, B\rangle$. Without loss we can assume that $B = C$. After conjugating with some elements in $\hat{G}$ we even can assume that $n = m = 0 \neq l \leqslant [M, C]$. Set $X = \langle A, B\rangle$ and let $M_0 = \langle l^X\rangle$. Then $M_0 = [M_0, A] + [M_0, B]$ and $[M_0, A] \cap [M_0, B] \subseteq C_M(G) = 1$ as follows by 6.5. Thus M_0 is an X-module satisfying the conditions 1, 2, and 3 of 1.3, see also [16, Lemma 3.2]. By Lemma 2 of [13], see also [15, 16], it follows that M_0 is a natural $SL(2, k)$-module and $X \simeq SL(2, k)$. Thus also in this case (m, A), (n, B), and (l, C) generate a group which modulo its center is isomorphic to a split extension of the natural $SL(2, k)$-module by $SL(2, k)$. This proves the proposition. ▮

Now Theorem 1.3 follows immediately from the above proposition, Theorem 1.2, and Lemma 6.4.

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